CSCAMM

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Velocity averaging, kinetic formulations and regularizing effects in conservation laws and related PDEs

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Loss of regularity

- Nonlinear conservation law:

$$\partial_t \rho + \partial_x A(\rho) = 0, \qquad a(\rho) := A'(\rho) \neq Const.$$

⊙ C^1 initial data $\rho(x, 0) = \rho_0(x)$

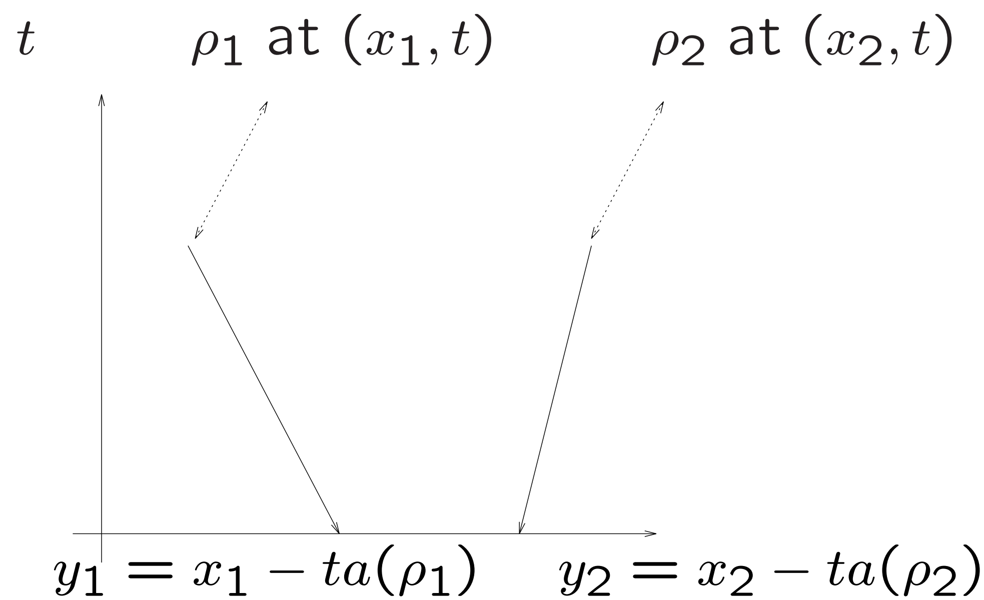
- Shock discontinuities:

C^1-smoothness breakdown at a finite time $t = t_c$.

- Generic: $t_c \sim -1/\partial_x a(\rho_0(x)) > 0$

Gain of regularity: $\partial_t\rho + \partial_x A(\rho) = 0, \qquad a(\rho) := A'(\rho)$

- Nonlinearity: $A(\rho)$ is convex $-$ $A''(\cdot) = a'(\cdot) \geq \alpha > 0$



- Entropy solution: $\dfrac{y_2 - y_1}{x_2 - x_1} \geq 0 \Rightarrow \dfrac{1}{t} \geq \dfrac{a(\rho_2) - a(\rho_1)}{x_2 - x_1} \geq \alpha \dfrac{\rho_2 - \rho_1}{x_2 - x_1}$

- Olienik condition: $\rho_x(t,x) \leq \frac{1}{\alpha t}$; dispersive effect

- Regularizing effect: $S(t) : L^\infty \to BV$; compactness, irreversibility, ...

  ⊙ Compensated compactness: Tartar $-$ $a'(\rho) \neq 0$ a.e.

Analysis of nonlinear conservation laws

- BV compactness — translation invariance; mostly 1D

- Compensated compactness — restricted to 1+1 D; but ...

$$\partial_t \rho + \partial_{x_1} A_1(\rho) + \partial_{x_2} A_2(\rho) = 0$$

⊙ Nonlinearity: (E & Engquist, Bagnerini, Rascle & ET):

If $meas\,\{v \mid \xi_1 a_1(v) + \xi_2 a_2(v) = 0\} = 0 \implies S(t)$ is compact

- Measure valued solutions – restricted to scalar eq's

- Averaging velocity lemma – kinetic formulation

Plan of talk

- Kinetic formulation
- Averaging lemma for first and second order eq's
- Regularizing effects:

⊙ nonlinear conservation laws
⊙ convection-(degenerate) diffusion
⊙ elliptic eq's

Joint work w/T. Tao

Second-order eq's and their kinetic formulations

$$\sum_{j=0}^{d} \partial_{x_j} A_j(\rho) - \sum_{j,k=1}^{d} \partial^2_{x_j x_k} B_{jk}(\rho) = S(\rho)$$

- ⊙ Conservation laws: $A_0(\rho) = \rho,\ B_{jk} \equiv 0$
- ⊙ Degenerate diffusion, convection-diffusion: $\{B'_{jk}\} \geq 0$
- ⊙ Elliptic: $A_j \equiv 0$

- On a proper notion of a (weak) solution

- Kinetic formulation: pseudo-Maxwellian $\chi_\rho(v) := \left\{ \begin{array}{cc} +1 & 0 < v < \rho \\ -1 & \rho < v < 0 \\ 0 & otherwise \end{array} \right\}$

- "macroscopic" $\rho(x) =$ average of its "microscopic" distribution: $\rho = \int_v \chi_\rho(v)dv$

- velocity averaging: $A(\rho) = \int_v \mathbf{a}(v)\chi_\rho(v)dv, \quad B(\rho) = \int_v \mathbf{b}(v)\chi_\rho(v)dv$:

$$\mathbf{a}(v) \cdot \nabla_x \chi_\rho - \Big\langle \mathbf{b}(v)\nabla_x, \nabla_x \Big\rangle \chi_\rho + S(v)\partial_v \chi_\rho =: \partial_v g(x,v) \longleftrightarrow \text{"}RHS\text{"}$$

- Entropy/renormalized ...solutions: $RHS = \partial_v m(x,v), m \in \mathcal{M}^+$

Entropy/renormalized ...solutions

$$\sum_{j=0}^{d} \partial_{x_j} A_j(\rho) - \sum_{j,k=1}^{d} \partial^2_{x_j x_k} B_{jk}(\rho) - S(\rho) = \overbrace{\text{``}\epsilon\Delta\rho\text{''}}^{\text{RHS}=0}\Big|_{\epsilon\downarrow 0}$$

Entropy/renormalized ...solutions

$$\left\langle \eta'(\rho_\epsilon),\ \sum_{j=0}^{d} \partial_{x_j} A_j(\rho_\epsilon) - \sum_{j,k=1}^{d} \partial^2_{x_j x_k} B_{jk}(\rho_\epsilon) - S(\rho_\epsilon) \right\rangle = \text{``}\left\langle \eta'(\rho_\epsilon), \epsilon \Delta \rho_\epsilon \right\rangle\text{''}\Big|_{\epsilon \downarrow 0}$$

$$\sum_{j=0}^{d} \partial_{x_j} \overbrace{A_j^\eta}^{\eta' \mathbf{a}_j =: \left(A_j^\eta\right)'}(\rho) - \sum_{j,k=1}^{d} \partial^2_{x_j x_k} \overbrace{B_{jk}^\eta}^{\eta' \mathbf{b}_{jk} =: \left(B_{jk}^\eta\right)'}(\rho) - \eta'(\rho) S(\rho) \overbrace{= \text{``}\epsilon \eta'(\rho) \Delta \rho\text{''}}^{\text{negative measure if } \eta'' > 0} \leq 0$$

$$\partial_v g(x, v) = \mathbf{a}(v) \cdot \nabla_x \chi_\rho - \left\langle \mathbf{b}(v) \nabla_x, \nabla_x \right\rangle \chi_\rho + S(v) \partial_v \chi_\rho$$

Entropy/renormalized ...solutions

$$\Big\langle \eta'(\rho_\epsilon),\ \sum_{j=0}^{d} \partial_{x_j} A_j(\rho_\epsilon) - \sum_{j,k=1}^{d} \partial^2_{x_j x_k} B_{jk}(\rho_\epsilon) - S(\rho_\epsilon) \Big\rangle = \text{“}\Big\langle \eta'(\rho_\epsilon), \epsilon\Delta\rho_\epsilon \Big\rangle\text{”}\Big|_{\epsilon\downarrow 0}$$

$$\overbrace{\sum_{j=0}^{d} \partial_{x_j} A^\eta_j(\rho)}^{\eta'\mathbf{a}_j =: \left(A^\eta_j\right)'} - \overbrace{\sum_{j,k=1}^{d} \partial^2_{x_j x_k} B^\eta_{jk}(\rho)}^{\eta'\mathbf{b}_{jk} =: \left(B^\eta_{jk}\right)'} - \eta'(\rho)S(\rho) \quad \overbrace{= \text{“}\epsilon\eta'(\rho)\Delta\rho\text{”}}^{\text{negative measure if } \eta''>0} \quad \leq 0$$

$$\eta'(v)\partial_v g(x,v) = \eta'(v)\mathbf{a}(v)\cdot\nabla_x\chi_\rho - \Big\langle \eta'(v)\mathbf{b}(v)\nabla_x, \nabla_x \Big\rangle \chi_\rho + \eta'(v)S(v)\partial_v\chi_\rho$$

- for all convex η's: $\displaystyle\int \eta'(v)\partial_v g(v)\,dv = \ldots$

$$\ldots \overbrace{\int \eta'(v)\mathbf{a}(v)\cdot\nabla_x\chi_\rho}^{\sum_j \partial_{x_j} A_j(\rho)} - \overbrace{\int \Big\langle \eta'(v)\mathbf{b}(v)\nabla_x, \nabla_x \Big\rangle \chi_\rho}^{\sum_{j,k} \partial^2_{x_j x_k} B^\eta_{j,k}(\rho)} + \overbrace{\int \eta'(v)S(v)\partial_v\chi_\rho}^{-\eta'(\rho)S(\rho)} \leq 0$$

$\Longrightarrow$ $g(v)$ is a nonnegative measure: $g(v) = m(v) \in \mathcal{M}^+$

$\odot$ measures entropy dissipation; supported on “shocks”

Kinetic examples: $\rho(x,v) \rightleftharpoons f(x,v) = \chi_{\rho(x)}(v)$

- Scalar conservation laws: $\rho_t + \nabla_x \cdot A(\rho) = 0$, $\mathbf{a}(\cdot) := A'(\cdot)$

⊙ Kinetic formulation – Lions, Perthame, ET:

$$\Big(\partial_t + \mathbf{a}(v)\cdot\nabla_x\Big) f(x,t,v) = \partial_v m(x,t,v), \quad m \in \mathcal{M}^+$$

- Convection-(degenerate) diffusion equations:

$$\rho_t + \nabla_x \cdot A(\rho) - tr\left\{\nabla_x \otimes \nabla_x B(\rho)\right\} = \partial_v m(x,t,v), \quad \mathbf{b}(\cdot) := B'(\cdot) \geq 0$$

⊙ Kinetic formulation (no uniqueness)

$$\Big(\partial_t + \mathbf{a}(v)\cdot\nabla_x - \langle \mathbf{b}(v)\nabla_x, \nabla_x\rangle\Big) f(x,t,v) = \partial_v m(x,t,v), \quad m \in \mathcal{M}^+$$

⊙ Renormalized solution of Chen-Perthame, Souganidis, Karlsen, ...

- Elliptic eq's: $-tr\left\{\nabla_x \otimes \nabla_x B(\rho)\right\} = S(\rho)$, $\mathbf{b} \geq 0$, $sgn(c)S(c) \geq 0, \ldots$

⊙ Kinetic formulation:

$$\Big(-\langle \mathbf{b}(v)\nabla_x, \nabla_x\rangle + S(v)\partial_v\Big) f(x,v) = \partial_v m(x,v), \quad m \in \mathcal{M}^+$$

Velocity Averaging: 1/3

$$\Big(\mathbf{a}(v)\cdot\nabla_x - \langle \mathbf{b}(v)\nabla_x, \nabla_x\rangle\Big)\chi_{\rho(x)}(v) = \partial_v m(x,v)$$

$$\mathcal{L}_k(\nabla_x, v) f(x,v) = RHS(x,v), \qquad k = 1, 2, \ldots,$$

- Hyperbolic regularity: $f(x,v)$ is as smooth as $RHS(x,v)$
- Averaging lemma: $\overline{f}(x) := \int f(x,v)dv$ is smoother

⋆ Averaging in the microscopic scale v:
leads to gain of smoothness in the macroscopic scale x

- First-order $\mathcal{L}$'s: Agoshkov, Perthame, Golse, Lions, Gerard, ...
  DiPerna-Lions-Meyer
  Jabin, DeVore-Petrova, ..., Golse-Saint-Raymond,
  Liu-Yu mixture lemma, Panov et. al.,...

  Brenier, Lions-Perthame-ET, Souganidis, ...

- Second-order $\mathcal{L}$'s....

Velocity averaging 2/3: $\mathcal{L}_k(\nabla_x, v)f = RHS(x,v)$

- $\widehat{f}(\xi,v) = \dfrac{\widehat{RHS}(\xi,v)}{\mathcal{L}_k(\xi,v)}$: gain of regularity whenever $\mathcal{L}_k(\xi,v) \neq 0$:

- Non-degeneracy $\Omega_\delta(\xi) := \left\{v : \ |\mathcal{L}_k(\xi,v)| < \delta\right\}$

$$\Big|\Omega_\delta(\xi)\Big| \lesssim \left(\tfrac{\delta}{J^\beta}\right)^\alpha, \ \ |\xi| \sim J$$

$$\overline{f}(x) \equiv \sum_{|\xi|\sim J} \underbrace{\mathcal{F}^{-1}\int_v 1_{\Omega_\delta(\xi)}(v)\widehat{f}(\xi,v)dv}_{} + \sum_{|\xi|\sim J} \underbrace{\mathcal{F}^{-1}\int_v 1_{\Omega^c_\delta(\xi)}(v)\frac{\widehat{RHS}(\xi,v)}{\mathcal{L}_k(\xi,v)}dv}_{}$$

$$= \qquad "\overline{\mathrm{b}}_J"(x) \qquad + \qquad "\overline{\mathrm{g}}_J"(x)$$

Velocity averaging 2/3: $\mathcal{L}_k(\nabla_x, v)f = RHS(x,v)$

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$$\overline{f}(x) \equiv \sum_{|\xi|\sim J} \underbrace{\mathcal{F}^{-1}\int_v \widetilde{1}_{\Omega_\delta(\xi)}(v)\widehat{f}(\xi,v)dv} + \sum_{|\xi|\sim J} \underbrace{\mathcal{F}^{-1}\int_v \widetilde{1}_{\Omega^c_\delta(\xi)}(v)\frac{\widehat{RHS}(\xi,v)}{\mathcal{L}_k(\xi,v)}dv}$$

$$= \qquad \text{"}\overline{\mathsf{b}}_J\text{"}(x) \qquad + \qquad \text{"}\overline{\mathsf{g}}_J\text{"}(x)$$

- Key point: $\mathcal{F}^{-1}\int_v \varphi\Big(\dfrac{\mathcal{L}(\xi,v)}{\delta}\Big)\widehat{f}(\xi,v)dv : L^p_x \to L^p_x$ independent of δ

$$\Rightarrow \|\overline{\mathsf{b}}_J\|_{L^p} \lesssim \|f\|_{L^p(x,v)} \left(\frac{\delta}{J^\beta}\right)^{\frac{\alpha}{p'}}, \qquad \|\overline{\mathsf{g}}_J\|_{W^k(L^q)} \lesssim \|RHS\|_{L^q(x,v)} \left(\frac{\delta}{J^\beta}\right)^{\frac{\alpha}{q'}-1}$$

$$f \in L^p,\ RHS \in L^q \implies \rho = \overline{f} \in W^s(L^1),\ s = s(\alpha,\beta,p,q) > 0$$

Velocity averaging: 3/3

$$\mathcal{L}(\nabla_x, v) f(x,v) = RHS; \quad deg(\mathcal{L}(\xi,\cdot)) \le k, \quad f \to \chi_\rho,\ RHS \to \partial_v m$$

- The truncation property: $\varphi(\frac{\mathcal{L}(\xi,\cdot)}{\delta}) : L^p_x \longrightarrow L^p_x$ independent of δ

- Non-degeneracy: $$meas_v\Big\{v \ \Big|\ |\mathcal{L}(\xi,v)| < \delta\Big\} \le Const.\Big(\frac{\delta}{J^\beta}\Big)^\alpha, \ \ |\xi| \sim J$$

$$f \in L^p_{|p=2}, RHS = \partial_v m, m \in L^q_{|q=1} \Rightarrow \overline{f} \in W^s(L^1), \quad s < \frac{(2\beta - k)\alpha}{2\alpha+1}$$

- If $deg(\mathcal{L}(\xi,\cdot)) = k \le 2$ then it satisfies the truncation property

1st-order: $\mathcal{L}(\xi,v) = \mathbf{a}(v)\cdot i\xi$ $(\beta = k = 1)$, $\overline{f} \in W^s(L^1)$, $\quad s_\alpha = \frac{\alpha}{2\alpha+1}$

2nd-order: $\mathcal{L}(\xi,v) = \langle \mathbf{b}(v)\xi,\xi\rangle$ $(\beta = k = 2)$, $\overline{f} \in W^s(L^1)$, $s_\alpha = \frac{2\alpha}{2\alpha+1}$

Nonlinear conservation laws 1/2 (Lions-Perthame-ET)

$$\rho_t + \nabla_x \cdot A(\rho) = 0$$

- Kruzkov's entropy pairs:

$$\eta(\rho; v) = |\rho - v|, \qquad A_j^\eta = sgn(\rho - v)\Big(A_j(\rho) - A_j(v)\Big):$$

$$m(t,x,v) := -\left[\partial_t \frac{\eta(\rho;v) - \eta(0;v)}{2} + \nabla_x \cdot \left(\frac{\mathbf{A}^\eta(\rho;v) - \mathbf{A}^\eta(0;v)}{2}\right)\right]$$

- Differentiate:

$$\partial_t \chi_{\rho(t,x)}(v) + \mathbf{a}(v)\cdot\nabla_x \chi_{\rho(t,x)}(v) = \partial_v m(t,x,v) \quad \text{in } \mathcal{D}'((0,\infty)\times \mathrm{IR}_x^d \times \mathrm{IR}_v)$$

- Nonlinearity: $meas_v\Big\{v \;\Big|\; |A'(v)\cdot\xi| < \delta\Big\} \lesssim \delta^\alpha, \quad \forall |\xi| = 1$

- Averaging: $\chi_\rho \in L^2$, $m \in \mathcal{M}$:

$$\Longrightarrow \quad \rho(x,t) = \int \chi_{\rho(x,t)}(v)dv \in W^{s_\alpha}(L^1), \qquad s_\alpha = \tfrac{\alpha}{2\alpha+1}$$

Nonlinear conservation laws 2/2

$$\rho_t + \nabla_x \cdot A(\rho) = 0, \qquad \boxed{meas_v\left\{v \;\middle|\; |A'(v)\cdot\xi| < \delta\right\} \lesssim \delta^\alpha}$$

$$\rho \in W^{s,1}, \;\; s = \frac{\alpha}{2\alpha+1}$$

- Examples of regularizing effects: $\rho \in W^s(L^1)$:

$$\partial_t \rho + (\sin(\rho))_{x_1} + (\frac{1}{3}\rho^3)_{x_2} = 0 \;\; \rightarrow \;\; s < \frac{1}{6}$$

$$|\tau + \xi_1 \cos(v) + \xi_2 v^2| < \delta \rightarrow \;\; \alpha = 1/4$$

$$\partial_t \rho + (|\rho|^m \rho)_{x_1} + (|\rho|^\ell \rho)_{x_2} = 0 \;\; \rightarrow \;\; s < \min\left\{\frac{1}{m+2}, \frac{1}{\ell+2}\right\}, \;\; m \neq \ell$$

Nonlinear conservation laws 2/2 (continued)

- Propagation of oscillations:

⊙ Lack of compactness: 2D Burgers $\rho_t + (\frac{\rho^2}{2})_{x_1} + (\frac{\rho^2}{2})_{x_2} = 0$:

⊙ linearized symbol $\xi_0 + \xi_1 v + \xi_2 v = 0, \quad \forall \xi_1 + \xi_2 = 0, \xi_0 = 0$

⊙ $\rho_0(x-y)$ are steady solution - oscillations persist along $x_1 - x_2 = \text{const.}$

⊙ Not BV optimal; but $\frac{1}{3}$ optimality w/$\alpha = 1$ - De Lellis & M. Westdickenberg

Nonlinear parabolic equations 1/4 (ET-Tao)

$$\partial_t\rho + \nabla_x \cdot A(\rho) - \sum \partial^2_{x_j x_k} B_{jk}(\rho) = 0, \;\; \mathbf{a} := A', \;\; \mathbf{b} := B' \geq 0$$

- Kinetic formulation for $f = \chi_\rho$:

$$\partial_t f + \mathbf{a}(v) \cdot \nabla_x f - \langle \mathbf{b}(v)\nabla_x, \nabla_x\rangle f = \partial_v m, \qquad m = m_A + m_B \in \mathcal{M}^+$$

- Non-degeneracy: $\left|\left\{v : \;\; |\tau + a(v)\cdot\xi| + |\langle b(v)\xi, \xi\rangle| \lesssim \delta\right\}\right| \lesssim \delta^\alpha$

⊙ Parabolic degeneracy ... $\lambda_1(v) \geq \lambda_2(v) \geq \ldots \lambda_N(v) \geq 0$

$$\left|\left\{v : \;\; |\lambda_N(v)| \leq \delta\right\}\right| \lesssim \delta^\alpha \Longrightarrow \rho(x,t) \in W^{\frac{2\alpha}{2\alpha+1}}(L^1), \quad t \geq \epsilon$$

Porous media: $\rho_t + \Delta(\rho^{n+1}) = 0, \;\; b(v) \sim v^n \longrightarrow \rho \in W^{\frac{2}{n+2}}(L^1)$

- Beyond the isotropic cases ...

Nonlinear Convection diffusion 2/4 (ET-Tao)

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\Big\{\frac{1}{\ell+1}\rho^{\ell+1}(t,x)\Big\} - \frac{\partial^2}{\partial x^2}\Big\{\frac{1}{n+1}\rho^{n+1}(t,x))\Big\} = 0,$$

has a regularizing effect of order s:

$$\forall t \geq \epsilon : \ \rho_0 \in L^\infty(\mathbb{R}_x) \mapsto \rho(t,\cdot) \in W^{s,1}_{loc}(\mathbb{R}_x)$$

- ⊙ If $n \leq \ell$ then degenerate diffusion dominates: $s = \dfrac{2}{n+2}$
- ⊙ If $n \geq 2\ell$ then nonlinear hyperbolicity dominates: $s = \dfrac{1}{\ell+2}$
- ⊙ Intermediate regularization $\ell < n < 2\ell$:

$$s = \frac{n + (2\ell - n)\zeta}{2n + 2(\ell - n)\zeta + n\ell}, \qquad \zeta = \frac{n}{\ell} - 1$$

Nonlinear parabolic equations 2/4 (continued)

- Study the degenerate set

$$\omega_{\mathcal{L}}(J;\delta) := |\Omega_{\mathcal{L}}(J;\delta)|, \quad \Omega_{\mathcal{L}}(J;\delta) := \left\{ v : \ |\mathcal{L}(\tau,\xi,v)| \leq \delta, \ \tau^2 + |\xi|^2 \sim J \right\}$$

$$\Omega_{\mathcal{L}}\Big(J;\delta\Big) := \left\{ v \ \middle| \ J|\tau + v^{\ell}\xi| + J^2|v|^n\xi^2 \leq \delta \right\}, \quad \tau^2 + \xi^2 = 1, \ J \gtrsim 1, \ \delta \lesssim 1.$$

⊙ case #1: dominated by the diffusive part, when $n \leq \ell$:

$$\Omega_{\mathcal{L}}(J;\delta) \subset \Omega_b := \left\{ v : \ |v|^n \leq \delta/J^2 \right\} \longrightarrow \omega_{\mathcal{L}}(J;\delta) \lesssim (\delta/J^2)^{1/n}$$

$\alpha = 1/n, \ \beta = 2$

⊙ case #2: dominated by the convective part, when $n \geq 2\ell$:

$$\Omega_{\mathcal{L}}(J;\delta) \subset \Omega_a := \{ v : \ |v|^{\ell} \lesssim \delta/J \} \longrightarrow \omega_{\mathcal{L}}(J;\delta) \lesssim (\delta/J)^{1/\ell}$$

$\alpha = 1/\ell, \ \beta = 1$

Nonlinear parabolic equations 2/4 (final)

⊙ intermediate cases #3: $\ell < n < 2\ell$ interpolate ...

• Bootstrap argument $\rho \in W^{\theta}(L^1)$:

$$\rho \in W^{s,1} \Longrightarrow \chi_\rho \in W^{s,1} \cap L^\infty \Longrightarrow \chi_\rho \in W^{s,2}, \quad s \mapsto (1-\theta)\frac{s}{2} + k\theta$$

$$\rho \in W^{s,1}, \quad s = \frac{2k\theta}{1+\theta}$$

⊙ L^p-initial data (nonlinear interpolation w/L^1-contraction)

$$S(t) : \rho_0 \in L^p \cap L^1 \mapsto W^{s,1}, \quad s < \frac{k\alpha}{(2\alpha+1)p'}, \ p > 1$$

Non-isotropic diffusion 3/4 (ET-Tao)

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x_1}\Big\{\frac{1}{\ell+1}\rho^{\ell+1}(t,x)\Big\} \;+\; \frac{\partial}{\partial x_2}\Big\{\frac{1}{m+1}\rho^{m+1}(t,x)\Big\} - \\ - \sum_{j,k=1}^{2}\frac{\partial^2}{\partial x_j \partial x_k}B_{jk}(\rho(t,x)) = 0$$

with non-degenerate diffusion, $B'(v) > |v|^n$:

- $W^{s,1}$-regularity, $\forall t \geq \epsilon$: $\rho_0 \in L^\infty(\mathrm{IR}_x) \mapsto \rho(t,\cdot) \in W^{s,1}_{loc}(\mathrm{IR}_x)$:

$$s < \begin{cases} \min\Big\{\dfrac{1}{\ell+2}, \dfrac{1}{m+2}\Big\} & \text{if } n > 2\max(\ell,m)+1 \text{ and } \ell \neq m \\ \dfrac{2}{n+2}, & \text{if } n < \min(\ell,m) \text{ or } \ell = m \end{cases}$$

Fully-degenerate case 4/4 (ET-Tao)

$$\frac{\partial}{\partial t}\rho(t,x) + \left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left\{\frac{1}{\ell+1}\rho^{\ell+1}(t,x)\right\} - \left(\frac{\partial^2}{\partial x_1^2}-2\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\partial^2}{\partial x_2^2}\right)\left\{\frac{1}{n+1}\rho^{n+1}(t,x)\right\}=0,$$

⊙ lack of regularity on hyperbolic and parabolic parts:

$$\frac{\partial}{\partial t}\rho(t,x) + \left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\rho = 0 : \qquad \rho_0(x-y)$$

$$\frac{\partial}{\partial t}\rho(t,x) - \left(\frac{\partial^2}{\partial x_1^2}-2\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\partial^2}{\partial x_2^2}\right)\rho = 0 : \qquad \rho_0(x+y)$$

Full equation admits $W^{s,1}$-regularizing effect of order $s=s(\ell,n)$:

if $n>2\ell$: $\quad s(\ell,n)=\dfrac{6}{2+2n-\ell}$

Degenerate elliptic equations

$$-\sum \partial^2_{x_j x_k} B_{jk}(\rho) = S(\rho),\ \mathbf{b} := B' \geq 0,\ sgn(c)S(c) \leq C_0$$

- Kinetic formulation for $f = \chi_\rho$:

$$\Big(- \langle \mathbf{b}(v)\nabla_x, \nabla_x \rangle + S(v)\partial_v \Big) f = \partial_v m, \qquad m \in \mathcal{M}^+$$

- Non-degeneracy: $\Big|\Omega_\delta\Big| \lesssim \delta^\alpha, \quad \Omega_\delta := \Big\{ v :\ |\langle b(v)\xi, \xi\rangle| \leq \delta \Big\}$

- Interior regularity of L^∞-solution $\rho \in \mathcal{D}'(\mathcal{G})$:

$$\rho(x) \in W^{s,1}_{loc}(\mathcal{G}_1), \quad s < \min\Big(\alpha, \frac{2\alpha}{2\alpha+1}\Big)$$

- Beyond the isotropic cases ...

Final notes

- Nonlinear conservation laws

⊙ Convergence on unstructured grids:
L^1 contraction w/no BV bound (Noelle, Westdickenberg, ...)

⊙ Compactness and nonlinearity in multiD case:
Compensated compactness vs. kinetic formulation

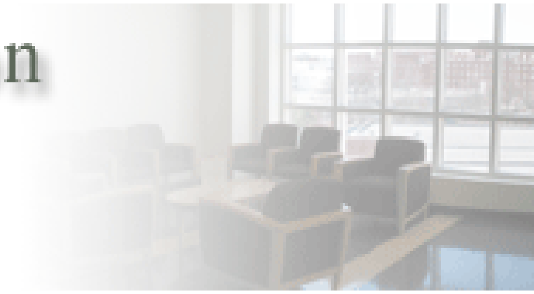
⊙ Convergence towards steady state: (E, Engquist, ...)

- Entropy/kinetic solution of elliptic equations

- Improved regularity: the first order case (s$>$ 1/3)
- Improved regularity: the second-oder dissipation

- What about equations of order$>$ 2?

- Systems: multivalued solutions,
(Lions, Perthame, Souganidis, ET, Brenier, ...)



THANK YOU